MATH2050C Selected Solutions to Assignment 3

Section 2.2 no. 10a, 11, 14b, 15d, 18a. Section 3.1 no. 5cd, 6a, 12, 14, 16, 17.

Section 2.2

(10a) Solve $|x-1| > |x+1|$.

Solution. Consider the cases (a) $x > 1$, (b) $x \in (-1, 1)$, (c) $x < -1$, (d) $x = 1$, (e) $x = -1$. In (a), the inequality becomes $x - 1 > x + 1$, no solution. In (b), we have $1 - x > 1 + x$, that is, $x < 0$. In (c), we have $1 - x > -x - 1$, that is, $x \in (-1, 1)$ always solution. In (d), no solution. In (e), $x = -1$ is a solution. Summing up the solution are all $x < 0$.

(18) When $a \leq b$, $\max\{a, b\} = b$ and $(a + b + |a - b|)/2 = (a + b - (a - b))/2 = b$. When $a > b$, $\max\{a, b\} = a$ and $(a+b+|a-b|)/2 = (a+b+(a-b))/2 = a$. Hence $\max\{a, b\} = (a+b+|a-b|)/2$ always holds.

Section 3.1

(5d) Show that $\lim_{n\to\infty} \frac{n^2-1}{2n^2+3} = \frac{1}{2}$. We have

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \left|\frac{-5}{2n^2+3}\right| < \frac{5}{2n^2} = \frac{5/2}{n^2} .$$

By Proposition 3.1, $\lim_{n\to\infty} \frac{n^2-1}{2n^2+3} = \frac{1}{2}$.

14 Show that $\lim_{n\to\infty} nb^n = 0$ for $b \in (0, 1)$. Write $b = 1/(1 + c), c > 0$. Using binomial expansion,

$$(1 + c)^n = 1 + nc + \frac{n(n-1)}{2}c^2 + \cdots + c^n > \frac{n(n-1)}{2}c^2 ,$$

that is, just keep the square term. It follows that

$$0 < nb_n \leq n \times 1/\frac{n(n-1)}{2}c^2 = \frac{2c^{-2}}{(n-1)} .$$

Clearly, $\frac{2c^{-2}}{n-1} \to 0$ as $n \to \infty$. By Proposition 3.1, we conclude $\lim_{n\to\infty} nb_n = 0$.

Supplementary Exercise

(1) Prove that $5^{2n} - 1$ can be divided by 8 for all $n \in \mathbb{N}$.

(2) Prove that for $a_1, a_2, \cdots, a_n \in \mathbb{R}$,

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n| .$$

(3) Prove the GM-AM Inequality: For $a_1, a_2, \cdots, a_n \geq 0$,

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{1}{n}(a_1 + a_2 + \cdots + a_n) , \quad n \geq 1,$$

and equality in the inequality holds iff all a_j's are equal.

Solution First show it is true for $n = 2^k, k \geq 1$. When $k = 1$, the inequality becomes

$$\frac{1}{2}(a+b) \geq ab, \ a, b \geq 0,$$

and equality holds iff $a = b$. This comes from the relation $(x-y)^2 > 0$ whenever $x \neq y$ (taking $a = \sqrt{x}$ and $b = \sqrt{y}$). Now assume the case $n = 2^k$ is true. We have

$$\begin{aligned}
a_1 + \cdots + a_{2^{k+1}} &= (a_1 + \cdots + a_{2^k}) + (a_{2^k+1} + \cdots + a_{2^{k+1}}) \\
&\geq 2\left[(a_1 + \cdots + a_{2^k})(a_{2^k+1} + \cdots + a_{2^{k+1}})\right]^{1/2} \\
&\geq 2\left[2^k(a_1 \cdots a_{2^k})^{1/2^k} \times 2^k(a_{2^k+1} \cdots a_{2^{k+1}})^{1/2^k}\right]^{1/2}) \quad \text{(induction hypothesis)} \\
&= 2^{k+1}(a_1 \cdots a_{2k+1})^{1/2^{k+1}} \ .
\end{aligned}$$

Also, equality holds iff all a_j's are equal. Now, for a general n. We fix some k such that $n < 2^k$ and consider $a_1, \cdots, a_n, a_{n+1}, \cdots, a_{2^k}$ where $a_{n+1} = \cdots = a_{2^k} = (a_1 + \cdots + a_n)/n$. Plugging this in the inequality for 2^k, after some computations, yields the inequality for n. Also equality holds iff all a_j's are equal.

(4) Show for each positive number a and $n \geq 2$, there is a unique positive number b satisfying $b^n = a$. Suggestion: Use Binomial Theorem.

Solution. Let $S = \{x > 0 : x^n < a\}$. Claim S is bounded from above: Pick some $N > a$ by Archimedean property, then $x^n < a$ implies $x^n < N \leq N^n$, so $N^n - x^n > 0$. By factorization $(N^{n-1} + N^{n-2}x + \cdots + x^{n-1})(N - x) > 0$. Since the first factor is positive, $N - x > 0$, that is, N is an upper bound of S. By order-completeness, $b = \sup S$ exists. Next we show that $b^n < a$ is impossible. Assume that it is true and we draw a contradiction. Letting $1 > \varepsilon > 0$ be small, we have

$$(b+\varepsilon)^n = b^n + \sum_{k=1}^{n} \binom{n}{k} b^{n-k}\varepsilon^k = b^n + \varepsilon \sum_{k=1}^{n} \binom{n}{k} b^{n-k}\varepsilon^{k-1} \ .$$

Using

$$\sum_{k=1}^{n} \binom{n}{k} b^{n-k}\varepsilon^{k-1} \leq \sum_{k=1}^{n} \binom{n}{k} b^{n-k} \equiv c \ ,$$

$(b+\varepsilon)^n \leq b^n + c\varepsilon$. If we choose ε satisfies $\varepsilon < (a - b^n)/c$, then $(b+\varepsilon)^n < b^n + c\varepsilon < a$, contradicting the fact that b is the supremum of S. A similar argument shows that $b^n > a$ is also impossible, thus leaves the only case $b^n = a$.

(5) For $a > 0$ and $m, n \in \mathbb{N}$, define $a^{m/n} = (a^{1/n})^m$ and $a^{-m/n} = a^{-(m/n)}$. Show that (a) $a^{m/n} = (a^m)^{1/n}$ and (b) $a^{r+t} = a^r a^t, r, t \in \mathbb{Q}$.

Solution. (a) By the definition of the n-th root, $[(a^m)^{1/n}]^n = a^m$. On the other hand,

$$(a^{m/n})^n = [(a^{1/n})^m]^n = (a^{1/n})^{mn} = (a^{1/n})^{nm} = [(a^{1/n})^n]^m = a^m.$$

We conclude that the n-th power of $(a^m)^{1/n}$ and $a^{m/n}$ are equal to a^m. From the uniqueness of roots $(a^m)^{1/n} = a^{m/n}$. (Here we have used the obvious fact $b^{mn} = (b^m)^n = b^{nm}$.)

(b) Let $r = m/n$ and $t = p/q$. Then

$$(a^r a^t)^{nq} = (a^{m/n} a^{p/q})^{nq} = (a^{m/n})^{nq}(a^{p/q})^{nq} = a^{mq}a^{pn}.$$

Also

$$(a^{r+t})^{nq} = (a^{m/n+p/q})^{nq} = a^{mq+pn} \ .$$

Using the formula $b^{k+l} = b^k b^l$ for $k, l \in \mathbb{Z}$, we have $a^{mq}a^{pn} = a^{mq+pn}$. Hence the nq-power of $a^r a^t$ and a^{r+t} are the same, so they are the same.